UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 1st FEBRUARY 2007

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

http://www.ukmt.org.uk

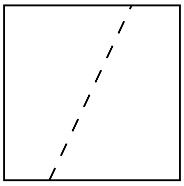
SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

1. D The date 18 years after 15 December 2005 will be 15 December 2023. Since a further 0.6 years, i.e. just over 7 months, will then elapse, the moon will next reach its highest point in the sky in 2024.

2. C There are forty dots in the 5×8 array and twenty of these are illuminated to form the letter 't'.

3. E Any line which passes though the centre of the square divides the square into two congruent shapes. An example is shown on the right.



4. E The fraction of the cake which Victoria eats is $\frac{2}{3} - \frac{1}{4}$, that is $\frac{5}{12}$.

5. D $(12340 + 12.34) \div 1234 = 12340 \div 1234 + 12.34 \div 1234 = 10 + 0.01 = 10.01$.

6. A The average of the 9 numbers is 223, so they are 219, 220, …, 226, 227.
(*Note that the difference between the largest and smallest of 9 consecutive whole numbers is 8, irrespective of the sum of the numbers*.)

7. B The product $= 1 \times 3 \times 4 \times 6 \times 7 \times 8 \times 9 \times (2 \times 5) \times 10$. The product of the first seven of these numbers is not a multiple of 10, so the original product is a multiple of 100, but not a multiple of 1000.

8. B The sum of the three numbers is $3x$, so $y + z = 2x$. Hence the mean of y and z is x.

9. B If there are f females and m males then $86 = 8f + 9m = 8(f + m) + m$. Now $86 = 8 \times 10 + 6 = 8 \times 9 + 14 = 8 \times 8 + 22 \ldots$. Only the first of these is feasible (because $f + m \geqslant m$). So $m = 6$ and $f = 4$.

10. C At each vertex of the triangle, four angles meet. In all, these twelve angles comprise six right angles, the three interior angles of the triangle and the three marked angles. So the sum of the marked angles $= 3 \times 360° - 6 \times 90° - 180° = 360°$.

11. A As the product of each pair has the same value, this value must be product of the smallest and largest numbers, that is 5×72. So the number which is paired with 10 is $(5 \times 72) \div 10$, that is 36.

12. A The formula which gives C in terms of d is $C = \pi d$, so the graph of C versus d is a straight line which passes through the origin.

13. E The area of the page inside the margins is 26 cm × 36 cm, that is 936 cm^2. So the percentage of the page occupied by the margins is $\frac{264}{1200} \times 100\%$, that is 22%.

14. A Both p and $-q$ are positive numbers. Hence $p + (-q)$ is the largest of the alternatives.

15. D Each interior angle of a regular pentagon is 108°, so $\angle SRQ = 108°$. As $SR = QR$, triangle SRQ is isosceles with $\angle RQS = \angle RSQ = \angle 36°$. Similarly, $\angle SRT = \angle STR = 36°$. So $\angle SUR = (180 - 2 \times 36)° = 108°$.
From the symmetry of the figure, $\angle PUR = \angle PUS = (360° - 108°) \div 2 = 126°$.

16. D In total, there are 12^3 cubes with edge length 1 cm. Each of these centimetre cubes has 12 edges, so the sum of the lengths of these edges is 12 cm. Therefore the total length of the edges of all the centimetre cubes is $12^3 \times 12$ cm, that is 12^4 cm.

17. C The two watches will next agree when Grannie's watch has gained twelve hours relative to Grandpa's watch. Each hour, Grannie's watch gains one hour relative to Grandpa's watch, so it will take 12 hours for this to happen. At this time, both watches will show a time of 6 o'clock.

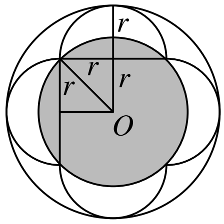
18. C From the second row we see that c, d, e are 1, 2, 4 in some order; and from the third column we see that $e > 2$. Hence $e = 4$ and we may now deduce that $g = 8$ and so $f = 5$. (*Although their values are not required, it is now also possible to deduce that* $c = 1$, $a = 3$, $b = 9$, $d = 2$.)

				Total
	a	b		12
	c	d	e	7
		f	g	13
Total	4	16	12	

19. E As each of the five given options is a member of the sequence, each is a multiple of 9. So we require the third factor of the number, that is the factor consisting of several 2s followed by a single 3, to be a multiple of 9 also. This is true if and only if the sum of its digits is a multiple of 9. For each number in the sequence, the number of 2s in its third factor equals the number of 0s in that number. The options given have 4, 6, 8, 10, 12 zeros, corresponding to their third factors having digital sums of 11, 15, 19, 23, 27 respectively. Of these, only 27 is a multiple of 9 so the correct answer is 20 000 000 000 007.

20. C Let O be the centre of the circle. Then $\angle POR = 90°$ as the angle subtended by an arc at the centre of a circle is twice the angle subtended by that arc at a point on the circumference of the circle. So triangle POR is an isosceles right-angled triangle with $PO = RO = 4$ cm. Let the length of PR be x cm. Then, by Pythagoras' Theorem, $x^2 = 4^2 + 4^2 = 2 \times 4^2$ and so $x = 4\sqrt{2}$.

21. B From the symmetry of the figure, the two circles must be concentric. Let their centre be O. Let the radius of the semicircles be r. Then the radius of the outer circle is $2r$ and, by Pythagoras' Theorem, the radius of the inner shaded circle is $\sqrt{r^2 + r^2}$, that is $\sqrt{2}r$.



So the radii of the two circles are in the ratio $\sqrt{2} : 2$, that is $1 : \sqrt{2}$, and hence the ratio of their areas is $1 : 2$.

22. A The table shows the number on the face in contact with the table at the various stages described and also the numbers on the three faces of the die visible from the viewpoint in the question.

In contact with the table	5	1	2	3	5
Facing South	3	3	3	5	4
Facing East	1	2	6	6	6
Facing Up	2	6	5	4	2

As the '6' face is now facing East, the '1' face will be facing West.

23. C Note that

$$\frac{n^2}{n+4} = \frac{n^2-16}{n+4} + \frac{16}{n+4} = \frac{(n+4)(n-4)}{n+4} + \frac{16}{n+4} = n - 4 + \frac{16}{n+4}, (n \neq -4).$$

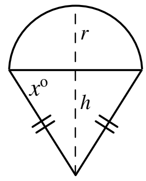
So when $n > 12$, the remainder when n^2 is divided by $n + 4$ is always 16. For $1 \leqslant n \leqslant 12$, the remainder when n^2 is divided by $n + 4$ is shown in the table below.

n	1	2	3	4	5	6	7	8	9	10	11	12
$n + 4$	5	6	7	8	9	10	11	12	13	14	15	16
remainder	1	4	2	0	7	6	5	4	3	2	1	0

So there are 9 different remainders, namely 0, 1, 2, 3, 4, 5, 6, 7, 16.

24. D Firstly, there are 12 unit squares which contain an even number. Every 2×2 square in the diagram has entries which consist of two odd numbers and two even numbers and hence have an even total. There are 16 of these. Each 3×3 square in the diagram, however, has entries which consist of five odd numbers and four even numbers (giving an odd total), or four odd numbers and five even numbers (giving an even total). There are 4 of the latter: those with 8, 12, 14 or 18 in the centre. Every 4×4 square in the diagram has entries which consist of eight odd numbers and eight even numbers and hence have an even total. There are 4 of these. Finally, the full 5×5 square contains 13 odd numbers and 12 even numbers, giving an odd total. So the required number is $12 + 16 + 4 + 4$, that is 36.

25. E Let the radius of the semicircle be r and let the perpendicular height of the isosceles triangle be h. Then $\tan x^\circ = \frac{h}{r}$.



Now the area of the semicircle $= \frac{1}{2}\pi r^2$, whilst the area of the triangle $= \frac{1}{2} \times 2r \times h = rh$. So $rh = \frac{1}{2}\pi r^2$, giving $\frac{h}{r} = \frac{\pi}{2}$.